Electric fields of a uniformly charged elliptical beam.

George Parzen

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Abstract

This paper presents results for the electric field due to a uniformly charged elliptical beam in the region outside the beam.

Introduction

This paper presents results for the electric field due to a uniformly charged elliptical beam outside the beam. Results for the field inside inside the beam are well known [1, 2] The beam being considered extends indefinitly in the z direction and has an elliptical boundary in x and y given by

$$x^2/a^2 + y^2/b^2 = 1 \tag{1}$$

The charge density, $\rho(x, y, z)$ is uniform within the elliptical boundary, zero outside the elliptical boundary, and does not depend on z. The results given below depend on the observation made by B. Houssais [3], that the result for the electric field of a gaussian charge distribution given by W. Kellog [1] as a one dimensional integral would hold for any elliptical charge distribution as defined below. This may be stated as follows. Let the charge distribution be given as

$$\rho(x, y, z) = \lambda n(x, y) \tag{2}$$

where λ is the charge per unit length and

$$\int dxdy\, n(x,y) = 1 \tag{3}$$

A charge distribution will be called elliptical if $n(x,y)$ can be written as

$$\begin{aligned} n(x,y) &= \hat{n}(T)/\pi ab \\ T &= x^2/a^2 + y^2/b^2 \end{aligned} \tag{4}$$

For the uniform elliptical beam , $\hat{n}(T)$ is given by

$$\begin{aligned} \hat{n}(T) &= 1, \ \ T \leq 1 \\ \hat{n}(T) &= 0, \ \ T > 1 \end{aligned} \tag{5}$$

For a Gaussian beam, $\hat{n}(T)$ is given by

$$\hat{n}(T) = exp(-T) \tag{6}$$

One can show, using Eq. 3, that $\hat{n}(T)$ obeys the equation

$$\int_0^\infty dT\, \hat{n}(T) = 1 \tag{7}$$

The generalization of the Kellog result for any elliptical beam is then

$$\begin{aligned} E_x &= 2\lambda \int_0^\infty dt \frac{\hat{n}(\hat{T})}{(a^2+t)^{3/2}(b^2+t)^{1/2}} \\ \hat{T} &= x^2/(a^2+t) + y^2/(b^2+t) \end{aligned} \tag{8}$$

A similar result, with a, b and x, y interchanged will give E_y

Electric fields for x,y inside the beam

As a first step, the fields inside a uniformly charged elliptical beam will be found using Eq. 8 .In this case, $\hat{T}$ is always ≤ 1 since for $t = 0$, $\hat{T} = x^2/a^2 + y^2/b^2$, which is ≤ 1 for x,y inside the beam, and decreases further for larger t. Eq. 8 then becomes

$$\begin{aligned} E_x &= 2\lambda x \int_0^\infty dt \frac{1}{(a^2+t)^{3/2}(b^2+t)^{1/2}} \\ \hat{T} &= x^2/(a^2+t) + y^2/(b^2+t) \end{aligned} \tag{9}$$

The integral in Eq. 9 can be done using the result

$$\int_{t_1}^{\infty} dt \frac{1}{(a^2+t)^{3/2}(b^2+t)^{1/2}} = 2\frac{1}{(a^2+t_1)^{1/2}} \frac{1}{(a^2+t_1)^{1/2}+(b^2+t_1)^{1/2}} \tag{10}$$

This gives

$$E_x = 4\lambda x \frac{1}{a(a+b)} \tag{11}$$

and a similar result for E_y with a and b interchanged and x replaced by y.

Electric fields outside the beam when $y = 0$

As the next step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the case when $y = 0$. The results in this case are simpler and the mathematics is easier to comprehend. In this case, $\hat{T}$ is > 1 for $t = 0$ since for $t = 0$, $\hat{T} = x^2/a^2 + y^2/b^2$, which is > 1 for x,y outside the beam. For larger t, $\hat{T}$ decreases and reaches the vaue of 1 at $t = t_1$, and at still larger t, $\hat{T}$ decreases further always remaining smaller than 1.The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$\begin{aligned}
E_x &= 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2+t)^{3/2}(b^2+t)^{1/2}} \\
\hat{T} &= x^2/(a^2+t) && (12) \\
t_1 &= x^2 - a^2 \\
y &= 0 && (13)
\end{aligned}$$

Using Eq. 10. one finds

$$\begin{aligned}
E_x &= 4\lambda \frac{1}{x + (x^2+b^2-a^2)^{1/2}} \\
E_y &= 0 && (14) \\
y &= 0
\end{aligned}$$

$E_{xx} = \partial E_x/\partial x$ is given by

$$E_{xx} = -\frac{E_x}{(x^2+b^2-a^2)^{1/2}} \tag{15}$$

Electric fields outside the beam when $y \neq 0$

As the final step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the general case. In this case, $\hat{T}$ is > 1 for $t = 0$ since for $t = 0$, $\hat{T} = x^2/a^2 + y^2/b^2$, which is > 1 for x,y outside the beam. For larger t, $\hat{T}$ decreases and reaches the vaue of 1 at $t = t_1$, and at still larger t, $\hat{T}$ decreases further always remaining smaller than 1.The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$\begin{aligned} E_x &= 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2+t)^{3/2}(b^2+t)^{1/2}} \\ x^2/(a^2+t_1) + y^2/(b^2+t_1) &= 1 \end{aligned} \tag{16}$$

t_1 is the positive root of the equation

$$x^2/(a^2+t_1) + y^2/(b^2+t_1) = 1 \tag{17}$$

The quadratic equation for t_1, Eq. 17, can be solved to give

$$\begin{aligned} t_1 &= (B^2/4 + C)^{1/2} + B/2 \\ B &= x^2 + y^2 - a^2 - b^2 \\ C &= x^2 b^2 + y^2 a^2 - a^2 b^2 \end{aligned} \tag{18}$$

Eq. 16 gives the result for E_x

$$E_x = 4\lambda x \frac{1}{(a^2+t_1)^{1/2}} \frac{1}{(a^2+t_1)^{1/2} + (b^2+t_1)^{1/2}} \tag{19}$$

and a similar result for E_y with a and b interchanged and x replaced by y.

It may be usefull to also have results for the derivatives of the fields, $E_{xx}, E_{yy}, E_{xy} = E_{yx}$, where $E_{xx} = \partial E_x/\partial x$, $E_{yy} = \partial E_y/\partial y$ and $E_{xy} = \partial E_x/\partial y$. E_{xx} is found using Eq. 16 for E_x

$$E_{xx} = \frac{E_x}{x} - 2\lambda x \frac{1}{(a^2+t_1)^{3/2}(b^2+t_1)^{1/2}} \frac{dt_1}{dx} \tag{20}$$

dt_1/dx can be found from Eq. 17 for t_1 as

$$\frac{dt_1}{dx} = 2x \frac{(a^2+t_1)(b^2+t_1)^2}{x^2(b^2+t_1)^2 + y^2(a^2+t_1)^2} \tag{21}$$

This gives for E_{xx}

$$E_{xx} = \frac{E_x}{x} - 4\lambda x^2 \frac{(a^2+t_1)^{-1/2}(b^2+t_1)^{3/2}}{x^2(b^2+t_1)^2 + y^2(a^2+t_1)^2} \quad (22)$$

E_{yy} and dt_1/dy can be found by interchanging x and y, and a and b. E_{xy} can be found in the same way as

$$E_{xy} = -4\lambda xy \frac{(a^2+t_1)^{1/2}(b^2+t_1)^{1/2}}{x^2(b^2+t_1)^2 + y^2(a^2+t_1)^2} \quad (23)$$

References

[1] W.Kellog, Foundations of Potential Theory, (Dover Publications,New York,1953), p. 192.

[2] L. Teng, Report ANLAD-59 (1963)

[3] F.J. Sacherer, PAC71, p.1105, (1971)